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1998 J. Phys. A: Math. Gen. 31 3481

(http://iopscience.iop.org/0305-4470/31/15/013)

J. Phys. A: Math. Gen. **31** (1998) 3481–3491. Printed in the UK PII: S0305-4470(98)90030-0

Exact two-body eigenstates in scalar quantum field theory

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Received 15 December 1997

Abstract. The scalar Yukawa (or Wick–Cutkosky) model, in which complex scalar fields, φ and ψ, interact via a real scalar field, χ, is reformulated by using covariant Green functions. It is shown that exact few particle eigenstates of the resulting truncated quantum field theory Hamiltonian can be obtained in the Feshbach–Villars formulation. Analytic solutions for the arbitrary mass two-body case are obtained for massless chion exchange in $(3+1)$ dimensions. The binding energy is found to increase more rapidly with strength of coupling than in the case of corresponding results obtained using the regular and light-cone ladder Bethe–Salpeter approximations.

1. Introduction

There are few models in quantum field theory (QFT), particularly in $(3+1)$ dimensions (three spatial dimensions plus time) for which exact solutions can be obtained. Those that exist usually have some contrived or unrealistic physical characteristics (e.g. [1]), or treat part of the theory (for example the mediating field) classically (e.g. [2] and [3]). Even then it is not often that analytic results for such things as two-body binding energies can be determined. In this paper we consider the scalar Yukawa model, in which complex scalar fields $\varphi(x)$ and $\psi(x)$, with masses m and M, respectively, interact via a mediating real scalar field $\chi(x)$, which may be massive or massless. As such, the model is a scalar analogue of quantum electrodynamics (QED), where electrons and muons interact electromagnetically. We show that exact two-body eigenstates of the QFTheoretic Hamiltonian for this model can be determined in the canonical equal-time formalism. Furthermore, we show that these eigenstates of the QFT Hamiltonian lead to relativistic two-body equations for which the eigenenergies can be obtained analytically for arbitrary values of the masses m and M of the constituent particles, at least in the case of massless mediating fields.

The model is based on the Lagrangian density ($\hbar = c = 1$)

$$\begin{aligned}\mathcal{L} = \partial^{\nu}\varphi^*(x)\partial_{\nu}\varphi(x) - m^2\varphi^*(x)\varphi(x) + \partial^{\nu}\psi^*(x)\partial_{\nu}\psi(x) - M^2\psi^*(x)\psi(x) \\ +\tfrac{1}{2}\partial^{\nu}\chi(x)\partial_{\nu}\chi(x) - \tfrac{1}{2}\mu^2\chi^2(x) - g\varphi^*(x)\varphi(x)\chi(x) - G\psi^*(x)\psi(x)\chi(x)\end{aligned} \quad (1)$$

where $\mu = 0$ for massless mediating fields. In that case ($\mu = 0$) this model is usually called the Wick–Cutkosky model.

We shall consider a reformulation of this theory in which the mediating chion field is partially eliminated by means of covariant Green functions, as discussed recently [4]. In addition, we shall use the Feshbach–Villars (FV) formulation [5] for the complex phion and

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psion fields, in the same manner as has been used recently to determine solutions of the $\lambda(\varphi^*\varphi)^2$ theory [6].

The fields φ, ψ and χ of the model (1) satisfy the equations

$$\partial^\nu\partial_\nu\varphi(x) + m^2\varphi(x) = -g\chi(x)\varphi(x) \tag{2}$$

$$\partial^\nu\partial_\nu\psi(x) + M^2\psi(x) = -G\chi(x)\psi(x) \tag{3}$$

and their conjugates, as well as

$$\partial^\nu\partial_\nu\chi(x) + \mu^2\chi(x) = \rho(x) \tag{4}$$

where $\rho(x) = -g\varphi^*(x)\varphi(x) - G\psi^*(x)\psi(x)$.

As is well known from classical electromagnetic theory, equation (4) has the formal solution

$$\chi(x) = \chi_0(x) + \int \mathrm{d}x'\, D(x - x')\rho(x') \tag{5}$$

where $\mathrm{d}x = \mathrm{d}^N x\, \mathrm{d}t$ in $(N + 1)$ dimensions, $\chi_0(x)$ satisfies the homogeneous (or free field) equation ((4) with $\rho = 0$), while $D(x - x')$ is a covariant Green function (or chion propagator, in QFTheoretic language), such that

$$(\partial^\nu\partial_\nu + \mu^2)D(x - x') = \delta^{N+1}(x - x'). \tag{6}$$

Equation (6) does not specify $D(x - x')$ uniquely since, for example, any solution of the homogeneous equation can be added to without invalidating (6). Boundary conditions based on physical considerations are used to pin down the form of D. Substitution of the formal solution (5) into equations (2) and (3) yields the 'reduced' equations

$$\partial^\nu\partial_\nu\varphi(x) + m^2\varphi(x) = -g\varphi(x)\chi_0(x) - g\varphi(x)\int \mathrm{d}x'\, D(x - x')\rho(x') \tag{7}$$

and

$$\partial^\nu\partial_\nu\psi(x) + M^2\psi(x) = -G\psi(x)\chi_0(x) - G\psi(x)\int \mathrm{d}x'\, D(x - x')\rho(x'). \tag{8}$$

These equations are derivable from the action principle $\delta\int \mathrm{d}x\, \mathcal{L} = 0$, corresponding to the Lagrangian density

$$\begin{aligned}\mathcal{L} = {} & \partial^\nu\varphi^*(x)\partial_\nu\varphi(x) - m^2\varphi^*(x)\varphi(x) - g\varphi^*(x)\varphi(x)\chi_0(x) \\ & +\partial^\nu\psi^*(x)\partial_\nu\psi(x) - M^2\psi^*(x)\psi(x) - G\psi^*(x)\psi(x)\chi_0(x) \\ & +\frac{1}{2}\int \mathrm{d}x'\, \rho(x)D(x - x')\rho(x')\end{aligned} \tag{9}$$

provided that $D(x - x') = D(x' - x)$.

QFTs based on (1) and (9) are equivalent in the sense that they lead to identical invariant matrix elements in various orders of covariant perturbation theory. The difference is that, in the formulation based on (9), the interaction term that contains the chion propagator $D(x - x')$ leads to Feynman diagrams that correspond to processes involving virtual chions only. On the other hand, the interaction term that contains χ_0 corresponds to Feynman diagrams that cannot be generated by the previous term, such as those with external (physical) chion lines.

The reformulated Lagrangian (9) contains two types of interactions: 'local' interactions of the particle densities $\varphi^*(x)\varphi(x)$ and $\psi^*(x)\psi(x)$ with the free mediating field $\chi_0(x)$, and the 'non-local' interaction in which the chion propagator appears explicitly. This may seem like a complication rather than a simplification of the theory based on (1). However, as we will show, the form (9) leads to a model for which exact eigenstates of the QFT Hamiltonian can be obtained.

2. Feshbach–Villars formulation

We rewrite this theory in the Feshbach–Villars (FV) formulation [5]. The reason for doing so is that this leads to a QFTheoretic Hamiltonian which is Schrödinger-like in form, for which exact eigensolutions can be readily written down. In the FV formulation, the field φ and its time-derivative $\dot{\varphi}$ are replaced by a two-component vector; the same is done for ψ and $\dot{\psi}$. These vectors are defined as

$$\phi = \begin{bmatrix} \phi_1 = \dfrac{1}{\sqrt{2m}}(m\varphi + \mathrm{i}\dot{\varphi}) \\ \phi_2 = \dfrac{1}{\sqrt{2m}}(m\varphi - \mathrm{i}\dot{\varphi}) \end{bmatrix} \qquad \Psi = \begin{bmatrix} \Psi_1 = \dfrac{1}{\sqrt{2M}}(M\psi + \mathrm{i}\dot{\psi}) \\ \Psi_2 = \dfrac{1}{\sqrt{2M}}(M\psi - \mathrm{i}\dot{\psi}) \end{bmatrix} \tag{10}$$

so that, for example, $2m\varphi^*\varphi = (\phi_1^* + \phi_2^*)(\phi_1 + \phi_2) = \phi^\dagger\eta\tau\phi$, where η and τ are the matrices

$$\eta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \tau = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}. \tag{11}$$

In the FV formulation the equation of motion (2) takes on the form

$$\mathrm{i}\dot{\phi} = -\frac{1}{2m}\nabla^2\tau\phi + m\eta\phi + \frac{g}{2m}\tau\phi\chi \tag{12}$$

or, upon using (5), the form

$$\mathrm{i}\dot{\phi} = -\frac{1}{2m}\nabla^2\tau\phi + m\eta\phi + \frac{g}{2m}\tau\phi\chi_0 + \frac{g}{2m}\tau\phi\int \mathrm{d}x'\, D(x - x')\rho(x') \tag{13}$$

where $\rho = -g\varphi^*\varphi - G\psi^*\psi = -(g/2m)\phi^\dagger\eta\tau\phi - (G/2M)\Psi^\dagger\eta\tau\Psi$. The equations for Ψ are the same as (12) and (13), but with ϕ replaced by Ψ, m by M and g by G.

Equation (13), and the similar equation for Ψ, are derivable from the Lagrangian density

$$\begin{aligned} \mathcal{L}_{\mathrm{FV}}(x) = {} & \mathrm{i}\phi^\dagger(x)\eta\dot{\phi}(x) - \frac{1}{2m}\nabla\bar{\phi}(x)\cdot\nabla\phi(x) - m\phi^\dagger(x)\phi(x) - \frac{g}{2m}\bar{\phi}(x)\phi(x)\chi_0(x) \\ & + \mathrm{i}\Psi^\dagger(x)\eta\dot{\Psi}(x) - \frac{1}{2M}\nabla\overline{\Psi}(x)\cdot\nabla\Psi(x) - M\Psi^\dagger(x)\Psi(x) \\ & - \frac{G}{2M}\overline{\Psi}(x)\Psi(x)\chi_0(x) + \frac{1}{2}\int \mathrm{d}x'\, \rho(x)D(x - x')\rho(x') \end{aligned} \tag{14}$$

where $\bar{\phi} = \phi^\dagger\eta\tau$ and $\overline{\Psi} = \Psi^\dagger\eta\tau$. Note that $\mathcal{L}$ of equation (9) is not identical to $\mathcal{L}_{\mathrm{FV}}$. Indeed $\mathcal{L} = \mathcal{L}_{\mathrm{FV}} + (\partial/\partial t)(\varphi^*\dot{\varphi} + \psi^*\dot{\psi})$. However, they lead to identical equations of motion ((7), (8) and (13)), and so are equivalent in this sense. Henceforth, we base our results on $\mathcal{L}_{\mathrm{FV}}$.

3. Quantization

The momenta corresponding to ϕ_1 and ϕ_2 are

$$p_{\phi_1} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_1} = \mathrm{i}\phi_1^* \qquad p_{\phi_2} = -\mathrm{i}\phi_2^* \tag{15}$$

and similarly for Ψ_i, that is, ϕ_i^* and Ψ_i^* are, in essence, the momenta conjugate to ϕ_i and Ψ_i, respectively. Thus, the Hamiltonian density is given by the expression

$$\begin{aligned} \mathcal{H}(x) = {} & \phi^\dagger(x)\eta\hat{h}_m(x)\phi(x) + \frac{g}{2m}\bar{\phi}(x)\phi(x)\chi_0(x) + \Psi^\dagger(x)\eta\hat{h}_M(x)\Psi(x) \\ & + \frac{G}{2M}\overline{\Psi}(x)\Psi(x)\chi_0(x) - \frac{1}{2}\int \mathrm{d}x'\, \rho(x)D(x - x')\rho(x') \end{aligned} \tag{16}$$

where $\hat{h}_m(x) = \tau(-(1/2m))\nabla^2 + m\eta$, and where we have suppressed terms like $\nabla \cdot (\bar{\phi}(x)\nabla\phi(x))$ that vanish upon integration and application of Gauss' theorem.

We use canonical equal time quantization, whereupon the non-vanishing commutation relations are

$$[\phi_a(\boldsymbol{x}, t), \phi_b^\dagger(\boldsymbol{y}, t)] = [\Psi_a(\boldsymbol{x}, t), \Psi_b^\dagger(\boldsymbol{y}, t)] = \eta_{ab}\delta^N(\boldsymbol{x} - \boldsymbol{y}) \qquad a, b = 1, 2 \tag{17}$$

where η_{ab} are elements of the η matrix (11). Using these commutation relations, the QFTheoretic Hamiltonian can be written as

$$H = \int \mathrm{d}^N x \, [\mathcal{H}_0(x) + \mathcal{H}_\chi(x) + \mathcal{H}_I(x)] \tag{18}$$

where (suppressing the Hamiltonian of the free chion field)

$$\mathcal{H}_0(x) = \phi^\dagger(x)\eta\hat{h}_m(x)\phi(x) + \Psi^\dagger(x)\eta\hat{h}_M(x)\Psi(x) \tag{19}$$

$$\mathcal{H}_\chi(x) = \frac{g}{2m}\bar{\phi}(x)\phi(x)\chi_0(x) + \frac{G}{2M}\overline{\Psi}(x)\Psi(x)\chi_0(x) \tag{20}$$

and

$$\begin{aligned}
\mathcal{H}_I(x) = &-\frac{g^2}{8m^2}\int \mathrm{d}x' \, D(x - x')\bar{\phi}(x)(\bar{\phi}(x')\phi(x'))\phi(x) \\
&-\frac{G^2}{8M^2}\int \mathrm{d}x' \, D(x - x')\overline{\Psi}(x)(\overline{\Psi}(x')\Psi(x'))\Psi(x) \\
&-\frac{gG}{8mM}\int \mathrm{d}x' \, D(x - x')\bar{\phi}(x)(\overline{\Psi}(x')\Psi(x'))\phi(x) \\
&-\frac{gG}{8mM}\int \mathrm{d}x' \, D(x - x')\overline{\Psi}(x)(\bar{\phi}(x')\phi(x'))\Psi(x)
\end{aligned} \tag{21}$$

and where we have used $\tau^2 = 0$ to re-order $\bar{\phi}(x)\phi(x)\bar{\phi}(x')\phi(x')$ as $\bar{\phi}(x)(\bar{\phi}(x')\phi(x'))\phi(x)$, etc, in the last step of (21). Note that no infinities are dropped upon performing this 'normal ordering', since none arise on account of the $\tau^2 = 0$ property.

As already mentioned, $\mathcal{H}_I$ contains the covariant chion propagator, hence in conventional covariant perturbation theory it leads to Feynman diagrams with internal chion lines. On the other hand, $\mathcal{H}_\chi$ corresponds to Feynman diagrams with external chions. However, we shall not pursue covariant perturbation theory in this work, and so shall not consider that approach further. Rather, we shall consider an approach that leads to some exact eigenstates of the Hamiltonian (18), but with $\mathcal{H}_\chi = 0$.

4. Truncated model

In what follows we shall consider the truncated model for which the term $\mathcal{H}_\chi$ in (18) is suppressed. Such a Hamiltonian is appropriate for describing systems for which there is no annihilation or decay into chions, or chion–phion/psion scattering.

In the Schrödinger picture we can take $t = 0$. Therefore, we shall use the notation that, say $\phi(\boldsymbol{x}, t = 0) = \phi(\boldsymbol{x})$, etc, for QFT operators. This allows us to express the interaction part of the Hamiltonian (21) as

$$\begin{aligned}
\mathcal{H}_I(\boldsymbol{x}) = &-\frac{g^2}{8m^2}\int \mathrm{d}^N x' \, G(\boldsymbol{x} - \boldsymbol{x}')\bar{\phi}(\boldsymbol{x})(\bar{\phi}(\boldsymbol{x}')\phi(\boldsymbol{x}'))\phi(\boldsymbol{x}) \\
&-\frac{G^2}{8M^2}\int \mathrm{d}^N x' \, G(\boldsymbol{x} - \boldsymbol{x}')\overline{\Psi}(\boldsymbol{x})(\overline{\Psi}(\boldsymbol{x}')\Psi(\boldsymbol{x}'))\Psi(\boldsymbol{x})
\end{aligned}$$

$$-\frac{gG}{8mM}\int \mathrm{d}^N x'\, G(\boldsymbol{x}-\boldsymbol{x}')\bar{\phi}(\boldsymbol{x})(\overline{\Psi}(\boldsymbol{x}')\Psi(\boldsymbol{x}'))\phi(\boldsymbol{x})$$
$$-\frac{gG}{8mM}\int \mathrm{d}^N x'\, G(\boldsymbol{x}-\boldsymbol{x}')\overline{\Psi}(\boldsymbol{x})(\bar{\phi}(\boldsymbol{x}')\phi(\boldsymbol{x}'))\Psi(\boldsymbol{x}) \tag{22}$$

where

$$G(\boldsymbol{x}-\boldsymbol{x}') = \int_{-\infty}^{\infty} D(x-x')\,\mathrm{d}t' = \frac{1}{(2\pi)^N}\int \mathrm{d}^N p\, \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\frac{1}{\boldsymbol{p}^2+\mu^2}. \tag{23}$$

Explicitly, for $N = 3$ this becomes

$$G(\boldsymbol{x}-\boldsymbol{x}') = \frac{1}{4\pi}\frac{\mathrm{e}^{-\mu|\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x}-\boldsymbol{x}'|} \tag{24}$$

for $N = 2$ it is

$$G(\boldsymbol{x}-\boldsymbol{x}') = \frac{1}{2\pi}K_0(\mu|\boldsymbol{x}-\boldsymbol{x}'|) \tag{25}$$

where $K_0(z)$ is the modified Bessel function, whereas for $N = 1$ it has the form

$$G(x-x') = \frac{1}{2\mu}\mathrm{e}^{-\mu|x-x'|}. \tag{26}$$

5. Empty vacuum and one-particle eigenstates

We define an empty vacuum state, $|\tilde{0}\rangle$, such that

$$\phi_a|\tilde{0}\rangle = \Psi_a|\tilde{0}\rangle = 0. \tag{27}$$

This is different from the conventional Dirac vacuum $|0\rangle$ (the 'filled negative energy sea' vacuum), which is annihilated by only the positive frequency part of φ and ψ and by the negative frequency parts of φ^* and ψ^*.

With the definition (27), the state defined as

$$|1_\phi\rangle = \int \mathrm{d}^N x\, \phi^\dagger(\boldsymbol{x})\eta f(\boldsymbol{x})|\tilde{0}\rangle \tag{28}$$

where $f(\boldsymbol{x})$ is a two-component vector, is an eigenstate of the truncated QFT Hamiltonian ($\mathcal{H}_\chi = 0$) with eigenvalue E_1 provided that the $f(\boldsymbol{x})$ is a solution of the equation

$$\hat{h}_m(\boldsymbol{x})f(\boldsymbol{x}) = E_1 f(\boldsymbol{x}). \tag{29}$$

This is just the free-particle Klein–Gordon (KG) equation for stationary states ($|1_\phi\rangle$ is insensitive to H_I). It has, of course, all the usual negative-energy 'pathologies' of the KG equation. The presence of negative-energy solutions is a consequence of the use of vacuum (27). However, that is the price that has to be paid in order to obtain *exact* eigenstates of the truncated Hamiltonian (equation (18) with $\mathcal{H}_\chi = 0$). A $|1_\Psi\rangle$ state can be obtained in a similar fashion. We shall refer to $|1_\phi\rangle$ and $|1_\Psi\rangle$ as a one-KG-particle state.

6. Two-particle eigenstates

We can define two-KG-particle states, analogously to (28)

$$|2_{\phi\Psi}\rangle = \int \mathrm{d}^N x\, \mathrm{d}^N y\, F_{ab}(\boldsymbol{x},\boldsymbol{y})\Psi_a^\dagger(\boldsymbol{x})\phi_b^\dagger(\boldsymbol{y})|\tilde{0}\rangle \tag{30}$$

where summation on repeated indices a and b is implied. This state is an eigenstate of the truncated QFT Hamiltonian ((18) with $\mathcal{H}_\chi = 0$) provided that the (2×2) coefficient matrix $F = [F_{ab}]$ is a solution of the two-body equation

$$\eta \hat{h}_M(\boldsymbol{x}) \eta F(\boldsymbol{x}, \boldsymbol{y}) + [\eta \hat{h}_m(\boldsymbol{y}) \eta F^{\mathrm{T}}(\boldsymbol{x}, \boldsymbol{y})]^{\mathrm{T}} + V(\boldsymbol{x} - \boldsymbol{y}) \tau^{\mathrm{T}} F(\boldsymbol{x}, \boldsymbol{y}) \tau = E_2 F(\boldsymbol{x}, \boldsymbol{y}) \tag{31}$$

where the superscript T stands for 'transpose'. The potential here is given by

$$V(\boldsymbol{x} - \boldsymbol{y}) = -\frac{gG}{4mM} G(\boldsymbol{x} - \boldsymbol{y}) \tag{32}$$

where $G(\boldsymbol{x} - \boldsymbol{y})$ is specified in equations (24)–(26).

Equation (31) is a relativistic two-body Klein–Gordon–Feshbach–Villars-like equation, with an attractive Yukawa interparticle interaction. If $V = 0$, then equation (31) has the solution $F(\boldsymbol{x}, \boldsymbol{y}) = g_1(\boldsymbol{x}) g_2^{\mathrm{T}}(\boldsymbol{y})$, where each $f_i(\boldsymbol{x}) = \eta g_i(\boldsymbol{x})$ is a solution of the free KG equation (29), with eigenenergy ε_i, where $E_2 = \varepsilon_1 + \varepsilon_2$, as would be expected.

In the rest frame, $\boldsymbol{P}_{\text{total}}|2_{\phi\Psi}\rangle = 0$, equation (31) simplifies to

$$\tilde{h}_M(\boldsymbol{r}) F(\boldsymbol{r}) + [\tilde{h}_m(\boldsymbol{r}) F^{\mathrm{T}}(\boldsymbol{r})]^{\mathrm{T}} + V(\boldsymbol{r}) \tau^{\mathrm{T}} F(\boldsymbol{r}) \tau = E_2 F(\boldsymbol{r}) \tag{33}$$

where

$$\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}, \tilde{h} = \eta \hat{h} \eta \qquad \text{and} \qquad V(\boldsymbol{r}) = -\frac{gG}{4mM} G(\boldsymbol{r})$$

in this case. It is useful to write out equation (33) in component form, with

$$F(\boldsymbol{r}) = \begin{bmatrix} s(\boldsymbol{r}) & t(\boldsymbol{r}) \\ u(\boldsymbol{r}) & v(\boldsymbol{r}) \end{bmatrix} \tag{34}$$

namely

$$\left(m + M - E_2 - \frac{1}{2m}\nabla^2 - \frac{1}{2M}\nabla^2 + V\right) s + \left(\frac{1}{2m}\nabla^2 - V\right) t + \left(\frac{1}{2M}\nabla^2 - V\right) u + Vv = 0 \tag{35}$$

$$\left(-\frac{1}{2m}\nabla^2 + V\right) s + \left(\frac{1}{2m}\nabla^2 - \frac{1}{2M}\nabla^2 + M - m - E_2 - V\right) t - Vu + \left(\frac{1}{2M}\nabla^2 + V\right) v = 0 \tag{36}$$

$$\left(-\frac{1}{2M}\nabla^2 + V\right) s - Vt + \left(\frac{1}{2M}\nabla^2 - \frac{1}{2m}\nabla^2 + m - M - E_2 - V\right) u + \left(\frac{1}{2m}\nabla^2 + V\right) v = 0 \tag{37}$$

and

$$Vs - \left(\frac{1}{2M}\nabla^2 + V\right) t - \left(\frac{1}{2m}\nabla^2 + V\right) u - \left(m + M + E_2 - \frac{1}{2m}\nabla^2 - \frac{1}{2M}\nabla^2 - V\right) v = 0. \tag{38}$$

These equations have positive-energy solutions of the type $E_2 = m + M + \cdots$, negative-energy solutions of the type $E_2 = -m - M + \cdots$, and 'mixed' type solutions with $E_2 = m - M + \cdots$ and $E_2 = M - m + \cdots$ (this is clear, for example, if $V = 0$ and the particles are at rest).

For the positive-energy solutions, if we write $E_2 = m + M + \epsilon$, then in the non-relativistic limit $|(\epsilon, V, p^2/m, p^2/M)v| \ll |mv|, |Mv|$ (and similarly for s, t and u), and so equations (36)–(38) show that t, u are small and v doubly-small components, by factors $O(\epsilon/m, \epsilon/M)$. Thereupon, equation (35) reduces to

$$-\frac{1}{2m}\nabla^2 s(\boldsymbol{r}) - \frac{1}{2M}\nabla^2 s(\boldsymbol{r}) + V(\boldsymbol{r})s(\boldsymbol{r}) = \epsilon s(\boldsymbol{r}) \tag{39}$$

which is the usual time-independent Schrödinger equation for the relative motion of two particles, of masses m and M, interacting through the potential $V(\boldsymbol{r})$. Similarly, in the non-relativistic limit, v is the large component for the negative-energy solutions (i.e. $E_2 = -(m + M + \epsilon)$, and $s \to v$, $V \to -V$ in (39)), while t is the large component for one set of mixed energy solutions and u is the large component for the other. This is obvious from the form of the free-particle solutions ($V = 0$), which are (with $p = \boldsymbol{p}$)

$$F(\boldsymbol{r}) = s_0 \begin{bmatrix} 1 & \left(\dfrac{p}{\omega+m}\right)^2 \\ \left(\dfrac{p}{\Omega+M}\right)^2 & \left(\dfrac{\omega-m}{\Omega+M}\right)^2 \end{bmatrix} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}} \xrightarrow[p\ll m,M]{} s_0 \begin{bmatrix} 1 & \left(\dfrac{p}{2m}\right)^2 \\ \left(\dfrac{p}{2M}\right)^2 & \left(\dfrac{p^2}{4mM}\right)^2 \end{bmatrix} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}} \tag{40}$$

for $E_2 = \omega + \Omega = \sqrt{p^2 + m^2} + \sqrt{p^2 + M^2}$

$$F(\boldsymbol{r}) = t_0 \begin{bmatrix} \left(\dfrac{p}{\omega+m}\right)^2 & 1 \\ \left(\dfrac{\omega-m}{\Omega+M}\right)^2 & \left(\dfrac{p}{\Omega+M}\right)^2 \end{bmatrix} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}} \xrightarrow[p\ll m,M]{} t_0 \begin{bmatrix} \left(\dfrac{p}{2m}\right)^2 & 1 \\ \left(\dfrac{p^2}{4mM}\right)^2 & \left(\dfrac{p}{2M}\right)^2 \end{bmatrix} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}} \tag{41}$$

for $E_2 = \Omega - \omega$

$$F(\boldsymbol{r}) = u_0 \begin{bmatrix} \left(\dfrac{p}{\Omega+M}\right)^2 & \left(\dfrac{\omega-m}{\Omega+M}\right)^2 \\ 1 & \left(\dfrac{p}{\omega+m}\right)^2 \end{bmatrix} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}} \xrightarrow[p\ll m,M]{} u_0 \begin{bmatrix} \left(\dfrac{p}{2M}\right)^2 & \left(\dfrac{p^2}{4mM}\right)^2 \\ 1 & \left(\dfrac{p}{2m}\right)^2 \end{bmatrix} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}} \tag{42}$$

for $E_2 = \omega - \Omega$, and

$$F(\boldsymbol{r}) = v_0 \begin{bmatrix} \left(\dfrac{\omega-m}{\Omega+M}\right)^2 & \left(\dfrac{p}{\Omega+M}\right)^2 \\ \left(\dfrac{p}{\omega+m}\right)^2 & 1 \end{bmatrix} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}} \xrightarrow[p\ll m,M]{} v_0 \begin{bmatrix} \left(\dfrac{p^2}{4mM}\right)^2 & \left(\dfrac{p}{2M}\right)^2 \\ \left(\dfrac{p}{2m}\right)^2 & 1 \end{bmatrix} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}} \tag{43}$$

for $E_2 = -\omega - \Omega$, and where s_0, t_0, u_0 and v_0 are constants.

In the one-body limit, $E_2 = M + \epsilon$, where $M \to \infty$, u and v vanish in equations (35)–(38), while s and t satisfy the following equation,

$$\left(-\frac{1}{2m}\nabla^2 + V\right)\tau f + m\eta f = \epsilon f \tag{44}$$

where $f = [s, -t]^{\mathrm{T}}$. This is recognized to be the usual one-particle Klein–Gordon equation (in Feshbach–Villars form) with *scalar* coupling. A similar result is obtained in the $m \to \infty$ limit. In short, the two-body equations (35)–(38) have the correct one-body limit.

Equations (35)–(38) can be reduced by taking suitable linear combinations, whereupon it follows that ($E = E_2$)

$$m(m - M - E)u = (m - M)(m + M - E)s + M(M - m - E)t \tag{45}$$

$$m(m + M + E)v = M(m + M - E)s + (m + M)(M - m - E)t \tag{46}$$

and

$$\begin{aligned}&[(M - m + E)^2(E^2 - (M + m)^2) + 8mMEV]s \\ &\qquad = [(M + m + E)^2(E^2 - (M - m)^2) + 8mMEV]t.\end{aligned} \tag{47}$$

It is easily verified that the free particle solutions (40)–(43), in particular, satisfy these relations. From these results it follows that s and t, and so u and v, can be expressed in terms of the single function w, namely

$$s = [(M + m + E)^2(E^2 - (M - m)^2) + 8mMEV]w \tag{48}$$

and

$$t = [(M - m + E)^2(E^2 - (M + m)^2) + 8MmEV]w \tag{49}$$

where w satisfies the following equation ($E \neq 0$)

$$-\nabla^2 w + \frac{2mM}{E}Vw = \frac{1}{4E^2}[(M - m)^2 - E^2)((M + m)^2 - E^2)]w. \tag{50}$$

Once (50) is solved for w, the components s, t, u, v of the matrix F follow from (45)–(49).

In the equal-mass case, $M = m$, equation (50), as indeed (45)–(49), reduce to results derived previously for a single complex scalar field interacting via a scalar mediating field [7]. Note that $t = u$ in the equal mass limit.

Equation (50) is form-identical to the Schrödinger equation, and so can be solved in the same manner as the latter for both bound and continuum states. In general, this has to be done numerically. In some cases, such as for $\mu = 0$ (massless chion exchange) in $(3 + 1)$, and for $\mu \neq 0$ in $(1 + 1)$, analytic solutions of Schrödinger's equation are known.

7. Two-body bound states in $(3 + 1)$ for massless chion exchange

We consider the solution of equation (50) for $N = 3$ and massless chion exchange (i.e. $\mu = 0$), in which case the interparticle potential is simply the Coulomb potential. In this case one can use the known hydrogenic solutions of the Schrödinger equation to obtain the solutions of equation (50). Thus, for the bound states we obtain the eigenenergy condition

$$[(M + m)^2 - E^2][(M - m)^2 - E^2] = -4m^2M^2\frac{\alpha^2}{n^2} \tag{51}$$

where $\alpha = gG/16\pi mM$, and $n = 1, 2, 3, \ldots$ is the principal quantum number. This yields the positive energy two-particle bound state spectrum

$$\begin{aligned}E &= \sqrt{M^2 + m^2 + 2Mm\sqrt{1 - \left(\frac{\alpha}{n}\right)^2}} \\ &= M + m - \frac{1}{2}m_r\left(\frac{\alpha}{n}\right)^2 - \frac{1}{8}m_r\left(1 + \frac{m_r}{m + M}\right)\left(\frac{\alpha}{n}\right)^4 + \cdots\end{aligned} \tag{52}$$

where $m_r = mM/(m+M)$. This, evidently, has the correct Rydberg non-relativistic (low-α) limit. We see that E decreases uniformly from $E = m + M$ at $\alpha = 0$ to $E = (M^2 + m^2)^{1/2}$ at the critical value of $(\alpha/n) = 1$, beyond which E ceases to be real, and the wavefunctions cease to be normalizable. This behaviour is similar to what occurs in the case of the bound state energy spectra of one-particle equations with a Coulomb potential. Note that the relativistic spectrum (52) retains the 'accidental' Coulomb degeneracy with respect to ℓ, which occurs also for the one-body Klein–Gordon equation with *scalar* coupling (44).

The two-particle bound-state wavefunctions corresponding to the eigenenergies (52) can be lifted similarly from the Schrödinger hydrogenic results. For example, the ground state wavefunction (unnormalized) corresponding to (52) with $n = 1$ is $w = \mathrm{e}^{-\beta r}$, where $\beta = mM\alpha/E$.

There are no negative-energy bound state solutions in the present case since the potential effectively reverses sign for the negative-energy case (this also happens in the case of one-particle relativistic equations, such as the Klein–Gordon–Coulomb and Dirac–Coulomb equations). However, equation (51) also has positive-energy solutions of the 'mixed' type, with

$$E = \sqrt{M^2 + m^2 - 2Mm\sqrt{1 - \left(\frac{\alpha}{n}\right)^2}} = |M - m| + \frac{1}{2}\left(\frac{mM}{|M-m|}\right)\left(\frac{\alpha}{n}\right)^2 + \frac{1}{8}\left(\frac{mM}{|M-m|}\right)\left(1 - \frac{mM}{(m-M)^2}\right)\left(\frac{\alpha}{n}\right)^4 + \cdots \tag{53}$$

where the expansion is valid for $m \neq M$ (for $m = M$ the expansion is given in [7]). These unrealistic solutions do not have a Rydberg non-relativistic limit, and arise because of the retention of negative-energy solutions in the present formalism. For these mixed-energy solutions E increases monotonically with increasing α from a value of $E = 0$ at $\alpha = 0$ to the value $E = \sqrt{M^2 + m^2}$ at $\alpha = n$. It is of interest to note that the positive-energy and mixed-energy solutions join smoothly at $\alpha = n$. Thus, for $0 \leqslant \alpha < \alpha_{\mathrm{c}} = n$, $E(\alpha)$ forms a continuous double-valued function, with the upper branch being the positive-energy solution and the lower branch being the mixed-energy solution. This behaviour occurs in other two-body equations with massless mediating fields, such as the Gross one-time reduction of the Bethe–Salpeter equation for the scalar Yukawa (Wick–Cutkosky) model [8].

The question arises how the present positive-energy bound state solutions compare to corresponding results obtained in other formulations of this model. To our knowledge there have not been many studies of this scalar Yukawa model for the case of arbitrary m and M for various strengths of the coupling. In particular, we know of no analytic results of type (51). In the equal-mass limit, and for $\mu/m = 0.15$, there has been a recent study of the two-body bound state spectrum using the ladder Bethe–Salpeter approximation, various 'one-time' reductions of the Bethe–Salpeter equation (with all ladder and crossed ladder diagrams), and the Feynman–Schwinger formalism [8]. The comparison of these results with those of the present equation (50), with $m = M$ and $\mu/m = 0.15$, is discussed in detail in [7] and shall not be repeated here, except to say that the present results are very similar to the Gross equation results (with ladder and crossed ladder diagrams, and retardation). As such, they predict much stronger binding than the ladder Bethe–Salpeter approximation, though not as strong as the results using the Feynman–Schwinger formulation.

For the present case of arbitrary masses of the constituent particles, Ji and Furnstahl [9] have computed the 1s two-body bound state spectrum in a light-front ladder Bethe–Salpeter formulation, for both massless and massive 'chion' exchange. Their results, for $m/M = 1/2$, are close to the ladder Bethe–Salpeter calculations of Zur Linden and Mitter [10], both for the massless and massive chion exchange (they give results for $\mu/M = 0, 0.1$ and 0.5). For $\mu = 0$ both sets of results [9, 10] give substantially weaker binding than the present analytic results (52), except at weak coupling, as happens also in the equal mass ($m = M$) case [7]. Specifically, the values of $\sqrt{B} = \sqrt{1 - E/(m+M)}$ for various values of $\alpha = g^2/(16\pi mM)$

are (α, $\sqrt{B}$ [9], $\sqrt{B}$ (52)): (0.2, 0.07, 0.0671), (0.4, 0.13, 0.1368), (0.6, 0.18, 0.2133), (0.8, 0.22, 0.3053), (0.9, 0.24, 0.3666), (1.0, 0.26, 0.5046), where the Ji and Furnstahl [9] results (second entry) have been read from their figure 2. Note that the present formalism gives real values of E, and so $\sqrt{B}$, only for $\alpha \leqslant 1$, which is not the case with the ladder Bethe–Salpeter results (regular [10] or light-front [9]).

Ji and Furnstahl do not give analytic expansions of $B(\alpha)$ for arbitrary values of m and M beyond the non-relativistic Rydberg results. However, it is clear from the equal mass case ($m = M$) [7] that our results do not agree with the ladder Bethe–Salpeter expressions beyond $O(\alpha^2)$. These latter have unusual α^3 and $\alpha^3 \ln \alpha$ terms [2, 11] which do not arise in the present formulation (cf equation (52)) or in conventional (canonical) Tamm–Dancoff-like calculations [12].

We expect that much the same kind of divergence occurs between the present and ladder Bethe–Salpeter predictions for the *massive* chion exchange case (this is evident from the $m = M$ results [7]). For $\mu \neq 0$, the interaction is given by a Yukawa (or screened Coulomb) potential and so equation (50) does not have analytic solutions, though numerical results can be obtained readily by solving the Schrödinger-like radial equation.

We shall not present any results in $(2 + 1)$ or $(1 + 1)$ in this paper, except to say that equation (50), with the exponential potential (26) that arises in $(1 + 1)$, is analytically solvable, and the bound state eigenvalues can be expressed as zeros of Bessel functions [7].

8. Concluding remarks

We have shown that the scalar Yukawa model, in which scalar particles of masses m and M interact via the exchange of quanta of mass μ (the 'chions') can be recast in a form such that *exact* two-body eigenstates of the QFTheoretic Hamiltonian, in the canonic equal-time formalism, can be determined for the case where there are no free (physical) quanta of the mediating chion field (i.e. only virtual chions). This is achieved by the partial elimination of the mediating field by means of Green functions, as well as by the use of the Feshbach–Villars formulation of scalar FT and the use of an 'empty' vacuum state. The use of the empty vacuum leads to the retention of negative-energy solutions, akin to those that arise in one-particle Klein–Gordon and Dirac equations.

Analytic solutions for the two-particle bound state eigenenergies were obtained for massless chion exchange in $(3+1)$ (equation (52)) for arbitrary values of m and M and any strength of the coupling (up to a 'critical' value). We compare our results with those obtained using the Bethe–Salpeter formalism (regular and light cone) in the *ladder* approximation, and find that, as the coupling strengthens, the present results give increasingly stronger binding than the ladder Bethe–Salpeter values. We point out that the present approach also leads to analytic two-body eigenenergies in $(1 + 1)$ dimensions for the massive chion exchange case.

Exact three (or more) particle eigenstates, analogous to equation (30), can be written down in an obvious way [7]. Of course, they lead to many-body generalizations of equation (31), and carry with them all the well known technical difficulties of a three- or more-body problem.

Acknowledgments

The financial support of the Natural Sciences and Engineering Research Council of Canada for this work is gratefully acknowledged.

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